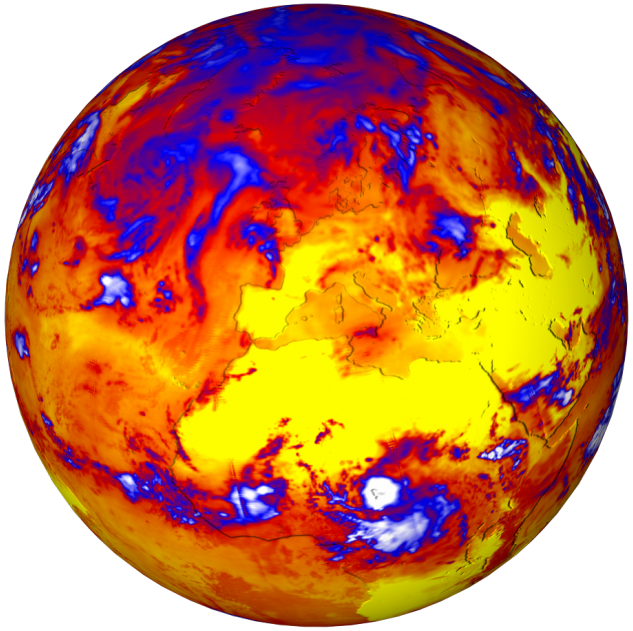
Fully Bayesian Inference for Spatial Extremes Using Hierarchical Extreme Value Processes

Ben Shaby

Joint work with Brian Reich



(Photo: NASA)

Outline

1. Introduction
2. Max-stable Processes
3. A Hierarchical Representation
4. Back to the Motivating Example
5. Discussion

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Motivating Example: Crops, heat shocks, and food security

From Shaby and Reich (2013):

Very high temperatures are detrimental to agriculture.

- Key concern: If one farm experiences a heat shock, it's a problem. If many farms do, it's a disaster.

Which leads to ...

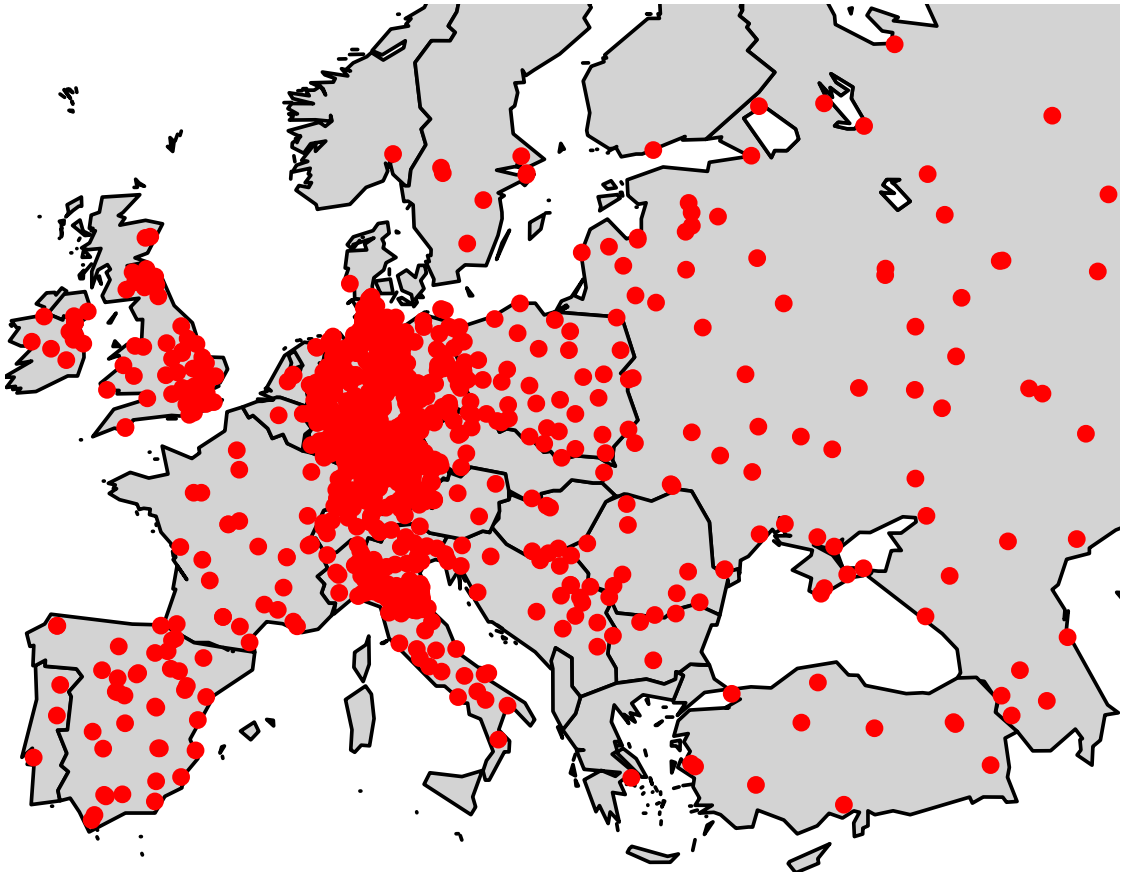
- Big question: For given values p and T, what is the probability that p percent of the agricultural land will experience a temperature higher than T in a given year?

And the follow-up:

- Is that probability changing?

What is the Data?

- Annual max temperatures at 985 monitoring locations for 1900–2011.



Goals and Strategy

We want to define a class of stochastic processes that

1. possess properties required by extreme value theory.
2. capture spatial dependence.
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We will get there by

1. truncating a spectral representation to give a finite collection of parameters.
2. building a spatial process using the truncated spectrum and latent variables.
3. conditioning on the spatial process to form a data likelihood.

* Bonus: this also allows us to do spatial prediction.

Advantages to a Bayesian Approach

In addition to the usual advantages,

- Models that assume unit Fréchet margins will require many marginal transformations.
- by definition we are data poor, and parameters are hard to estimate (especially ξ)!
  $\rightarrow$ Use priors to trade space for time.
- answers to the interesting questions are often probability statements about complicated transformations of unknown parameters.
- likelihoods of existing models are intractable.
  $\rightarrow$ Build process models conditionally.

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Max-stable processes

Max stability characterizes extreme value processes.

- Pointwise maxima of stochastic processes can only converge to max-stable processes.
- In this sense the only asymptotically justified class of maximal processes are max-stable.
- Can we build spatial processes that are max-stable?

A Spectral Representation

There is a more suggestive way to characterize extreme value processes (Schlather, 2002).

Let

- $\{H_j\}_{j=1}^{\infty}$ be a Poisson process on $\mathbb{R}_+$ with intensity measure dh/h^2.
- $\{W_j'(\mathbf{s})\}_{j=1}^{\infty}$ be independent copies of a stationary process on $\mathcal{D} \subset \mathbb{R}^d$, and $W_j(\mathbf{s}) = \max\{0, W_j'(\mathbf{s})\}$ for $j = 1, \ldots, \infty$ such that $\mathsf{E}[W_j(\mathbf{s})] = 1$ for all $\mathbf{s}$.

Then

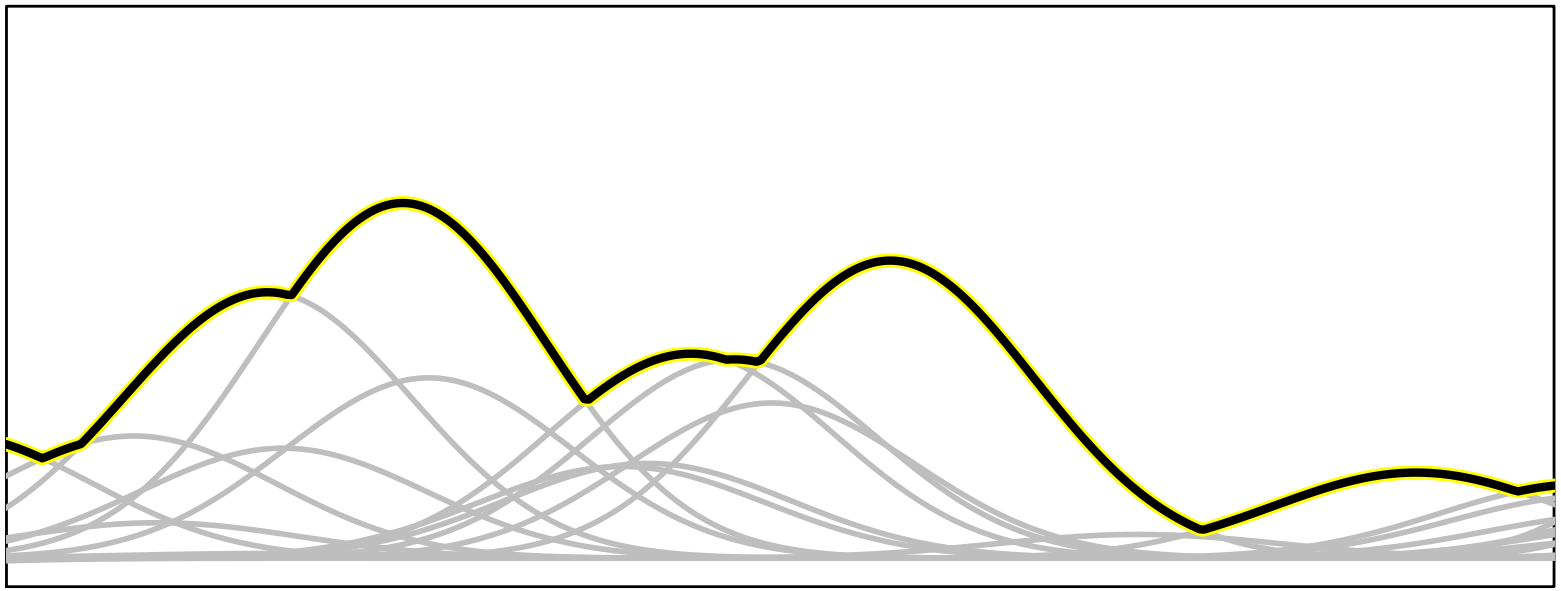
$$X(\mathbf{s}) = \max_j \{H_j W_j(\mathbf{s})\}$$

is a stationary max-stable process on $\mathcal{D}$ with unit Fréchet margins.

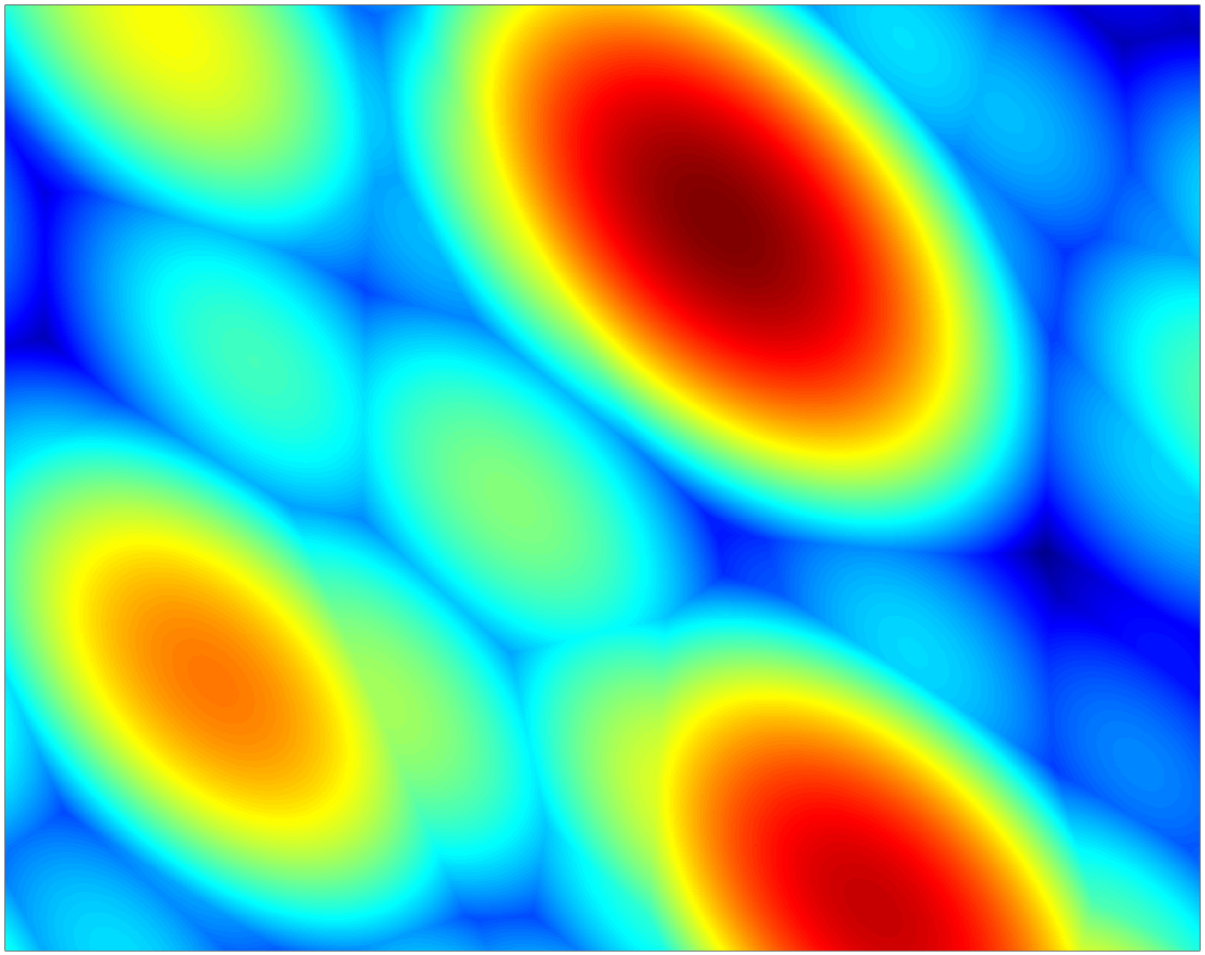
Example: Gaussian Extreme Value Process (GEVP)

To construct the "Smith process" (Smith, 1990), let

- $W_j(\mathbf{s}) = \varphi(\mathbf{s}; \mu_j, \mathbf{\Sigma})$ are normal kernels with mean μ_j and covariance matrix Σ, $j = 1, \ldots, \infty$,
- $\{\mu_j\}_{j=1}^{\infty}$ is a homogeneous Poisson process on $\mathcal{D}$.



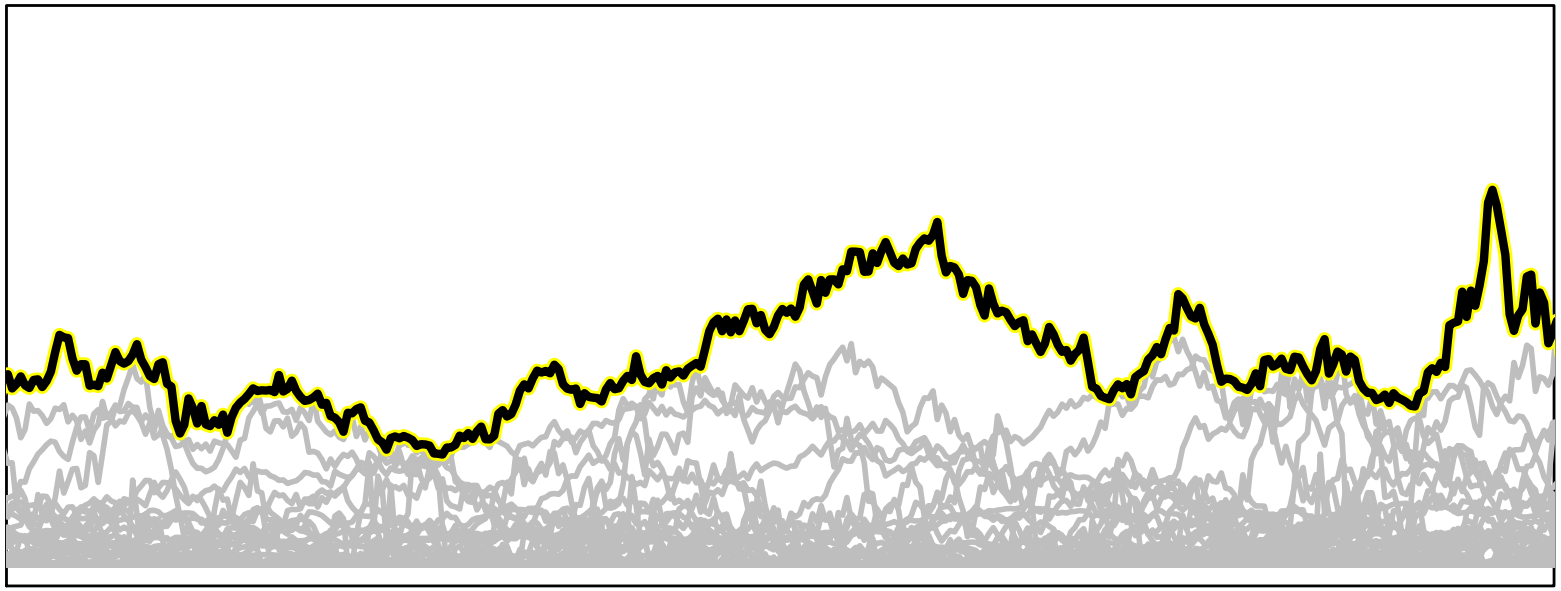
Example: Gaussian Extreme Value Process



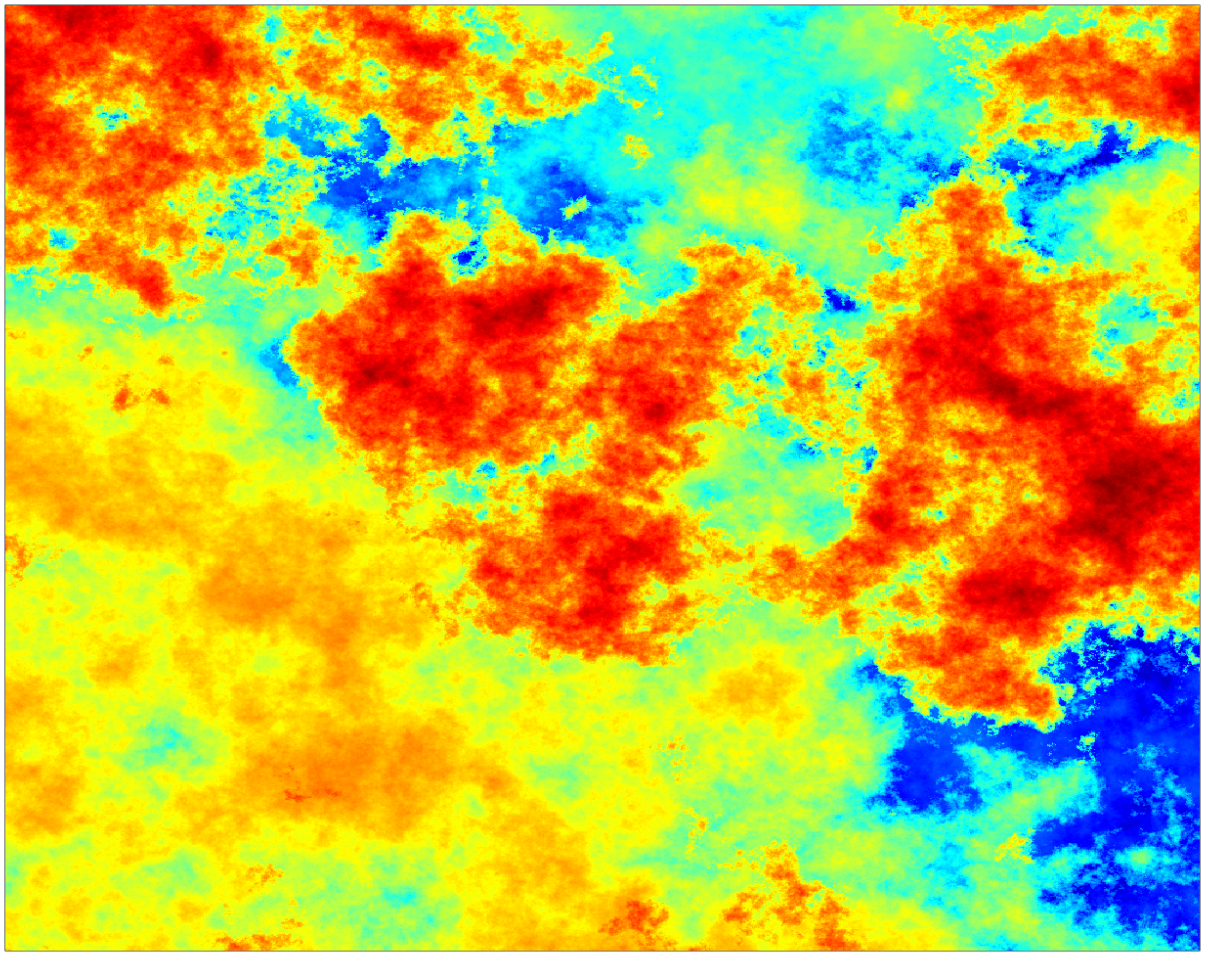
Example: Extremal Gaussian Process

To construct the "Schlather process" (Schlather, 2002), let

- $\{W_j'(\mathbf{s})\}_{j=1}^{\infty}$, be independent copies of a stationary Gaussian process with correlation function ρ, scaled such that $\mathrm{E}[\max\{0, W_j'(\mathbf{s})\}] = 1$.



Example: Extremal Gaussian Process



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Truncating the Spectral Representation

$$X(\mathbf{s}) = \max\{H_1 W_1(\mathbf{s}), H_2 W_2(\mathbf{s}), \ldots\}$$

- The first step is deciding that maybe we don't need an *infinite* number of W_js.
- Simply truncating this spectrum does not yield a likelihood.
- We seek a model with a finite number of components that
  1. is max-stable.
  2. gives a tractable likelihood.
- One key to producing a likelihood will be using latent positive stable variables (Fougères et al., 2009; Stephenson, 2009).

Positive Stable Random Variables

Let $A \sim \mathsf{PS}(\alpha)$ denote a positive stable random variable with characteristic exponent $\alpha \in (0, 1)$.

Fact to keep in mind about $\mathsf{PS}(\alpha)$:

- As $\alpha \to 1$, $f_A(a \,|\, \alpha) \to \delta(1)$

Truncated Spectrum with Latent PS(α) Random Variables

1. First, choose a large but finite number L "basis functions" $W_1(\mathbf{s}), \ldots, W_L(\mathbf{s})$.
2. Normalize each such that $W_{w,l}(\mathbf{s}) = W_l(\mathbf{s}) / \sum_{j=1}^{L} W_j(\mathbf{s})$. (replaces the moment condition)
3. Now define the spatial effect

$$X^*(\mathbf{s}) = \Big(\sum_{l=1}^{L} A_l W_{w,l}(\mathbf{s})^{1/\alpha} \Big)^{\alpha},$$

where $A_l \sim$ PS(α) independently for $l = 1, \ldots, L$.

The Role of α in $X^*(\mathbf{s})$

$$X^*(\mathbf{s}) = \Big(\sum_{l=1}^{L} A_l W_{w,l}(\mathbf{s})^{1/\alpha} \Big)^{\alpha},$$

- As $\alpha \to 0$,
  - $\left(\sum_{l=1}^{L} x^{1/\alpha} \right)^{\alpha} \to \max_l \{x_l\}$
  - so $X^*(\mathbf{s})$ becomes the max of randomly scaled spatial processes.

- As $\alpha \to 1$,
  - $A_l \xrightarrow{\mathcal{D}} 1$
  - so $X^*(\mathbf{s})$ becomes the constant 1.

Properties of $X^*(\mathbf{s})$

Let $U(\mathbf{s}) \overset{\text{iid}}{\sim} \text{GEV}(1, 1, 1)$.

Fact 1

$U(\mathbf{s})^\alpha X^*(\mathbf{s})$ *is max-stable.*

Fact 2

$U(\mathbf{s})^\alpha X^*(\mathbf{s})$ *has unit Fréchet margins for all* $\mathbf{s} \in \mathcal{D}$.

The Object $U(\mathbf{s})^{\alpha} X^{*}(\mathbf{s})$

$U(\mathbf{s})^{\alpha} X^{*}(\mathbf{s})$ is max-stable with unit Fréchet margins.

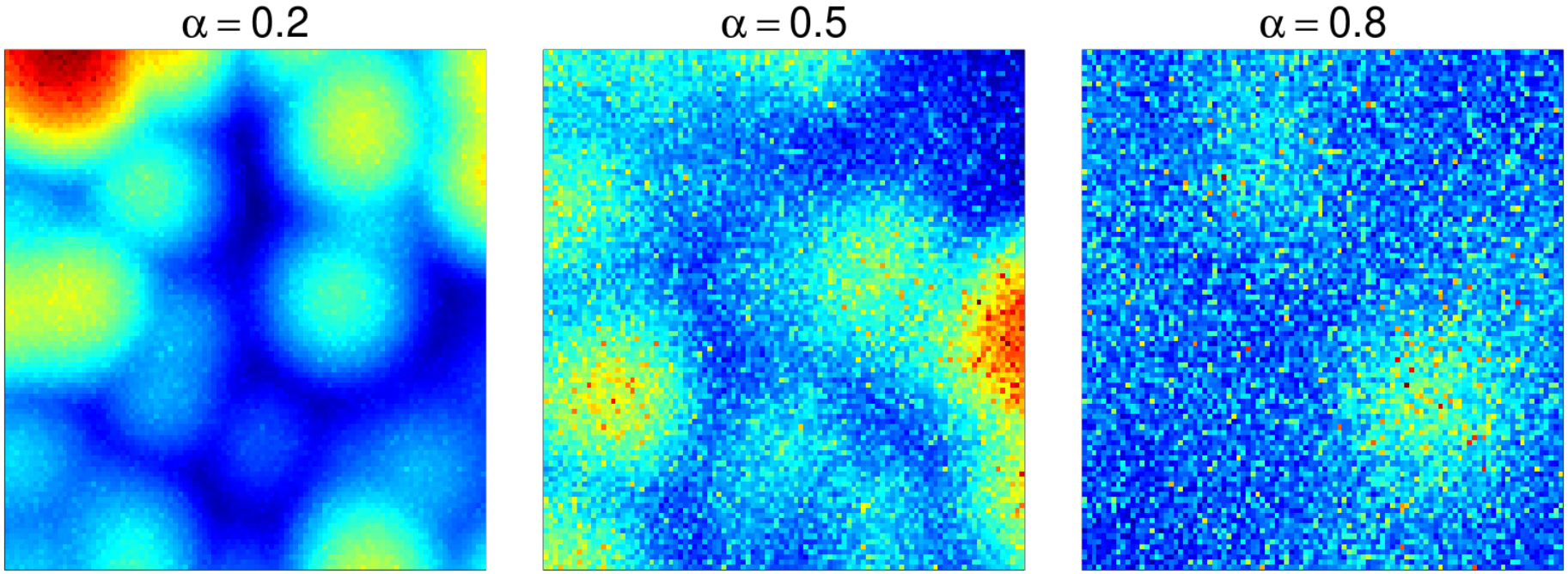
- $U(\mathbf{s}) \overset{\text{iid}}{\sim} \text{GEV}(1, 1, 1)$ is the extremal "nugget effect."
- This expression makes explicit that α trades off spatial with non-spatial variability.
- We know how to transform from $\text{GEV}(1, 1, 1)$ to $\text{GEV}(\mu, \sigma, \xi)$.
  - $\rightarrow$ Gives us a likelihood!

Example: The Hierarchical Kernel Extreme Value Process (HKEVP)

From Reich and Shaby (2013):

1. Choose a large but finite number L of fixed knots, $\mathbf{v}_1, ..., \mathbf{v}_L$.
2. Choose $W_1(\mathbf{s}), \ldots, W_L(\mathbf{s})$, Gaussian kernels centered at the knots to construct $X^*(\mathbf{s}) = \left(\sum_{l=1}^{L} A_l W_{w,l}(\mathbf{s}|\mathbf{v}_l)^{1/\alpha} \right)^{\alpha}$.

The effect of α



Fact 3

For equally spaced knots, the HKEVP converges in distribution to the GEVP as $\alpha \to 0$ and $L \to \infty$.

Example: The Hierarchical Log-Gaussian Extreme Value Process

Cribbed from from Davison et al. (2012):

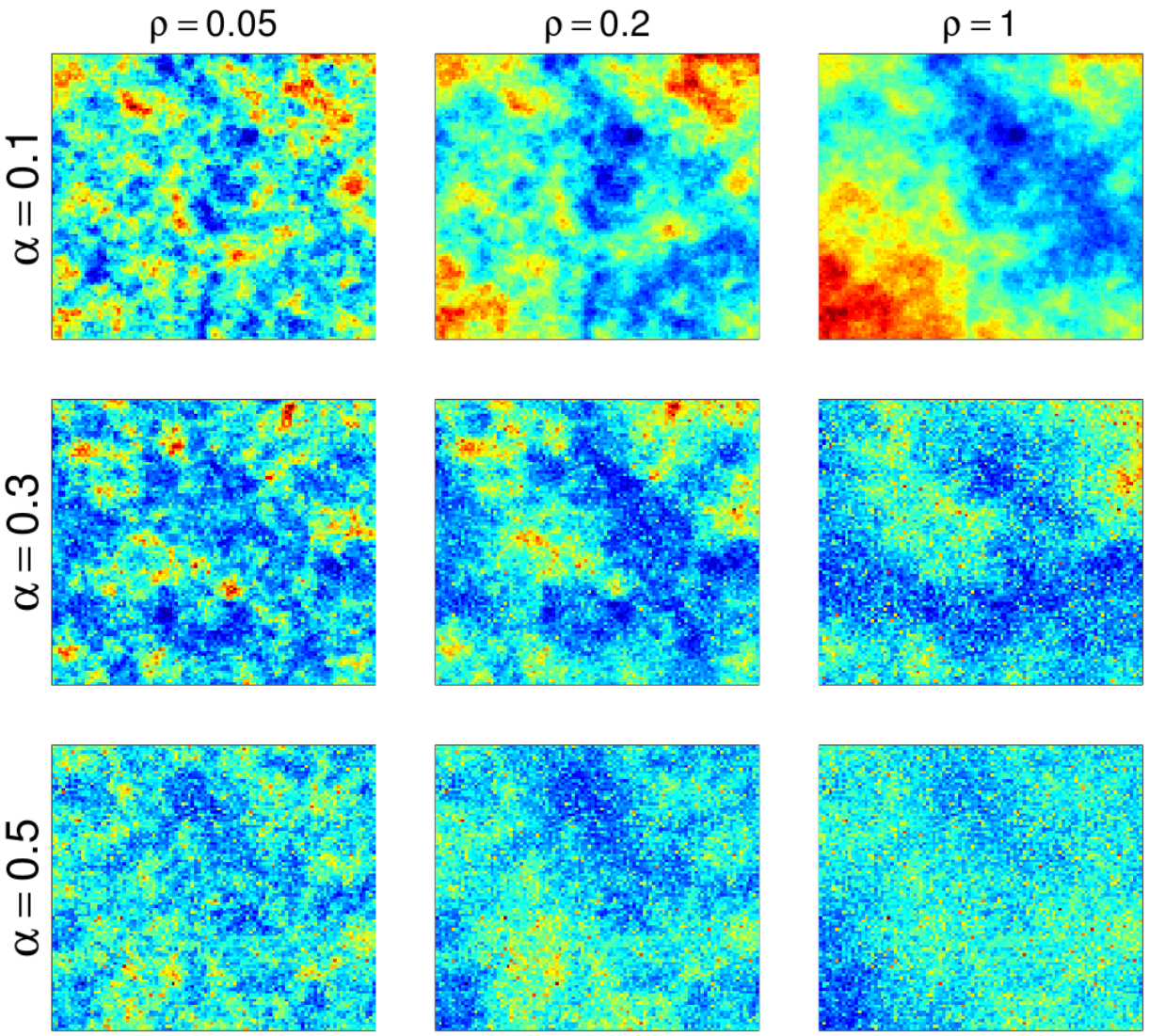
1. Choose $W_1(\mathbf{s}), \ldots, W_L(\mathbf{s})$ as independent copies of log Gaussian processes, with

$$\{W_l(\mathbf{s} \,|\, \sigma_W, \boldsymbol{\theta}_W)\}_{l=1}^{L} = \exp\{\sigma_W \varepsilon_l(\mathbf{s} \,|\, \boldsymbol{\theta}_W) - \sigma_W^2/2\}.$$

   - $\varepsilon_l(\mathbf{s} \,|\, \boldsymbol{\theta}_W)$ are standard Gaussian processes with covariance parameters $\boldsymbol{\theta}_W$.

2. Scale appropriately to meet sum-to-one constraint, and construct $X^*(\mathbf{s}) = \left(\sum_{l=1}^{L} A_l W_{w,l}(\mathbf{s})^{1/\alpha} \right)^{\alpha}$ as before.

The effect of α and the Range Rarameter



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Data Locations

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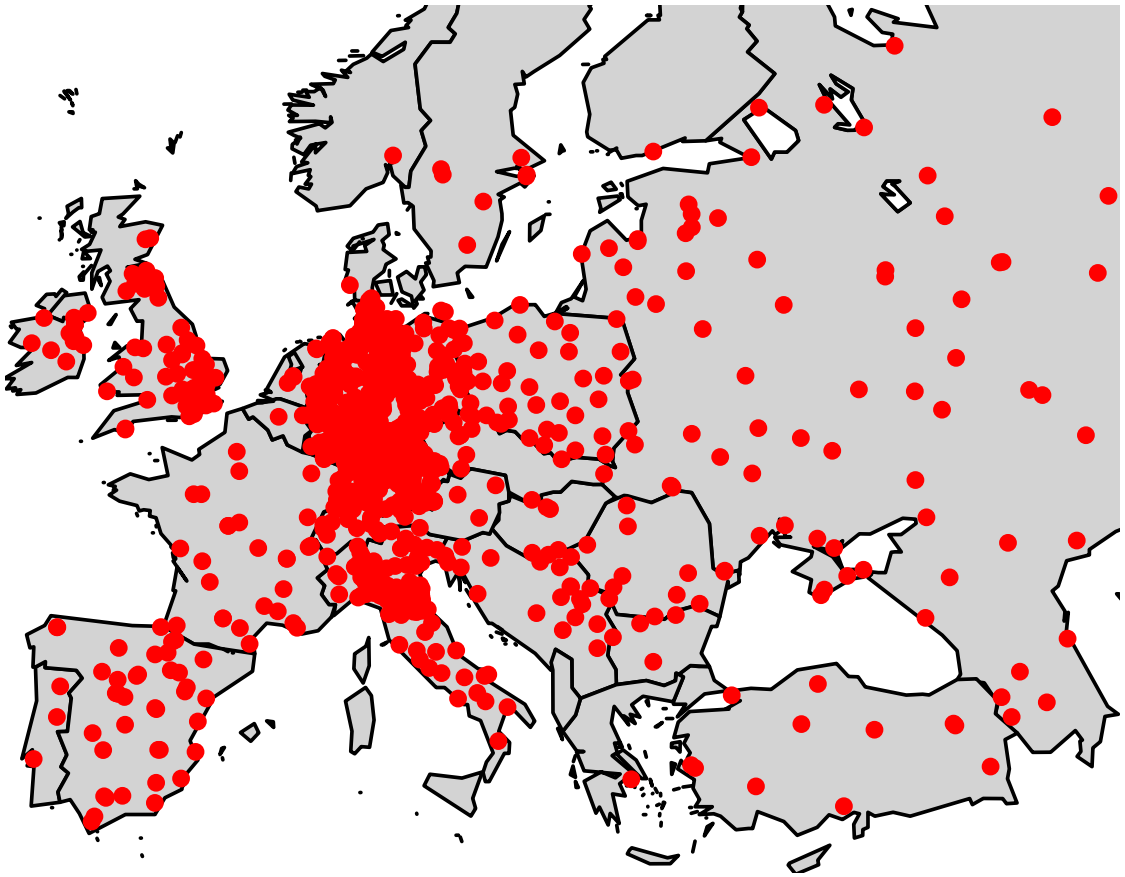
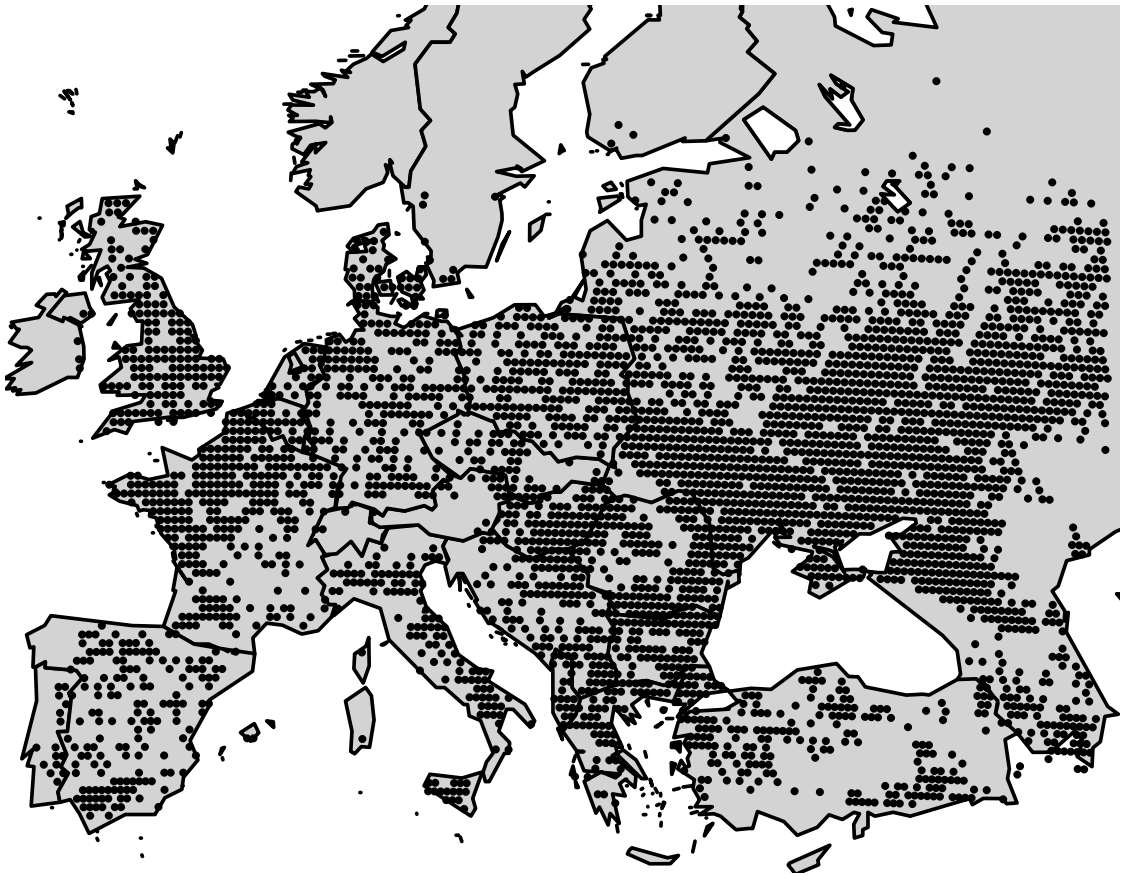
- Extracted years 1900–2011 from 985 monitoring locations.

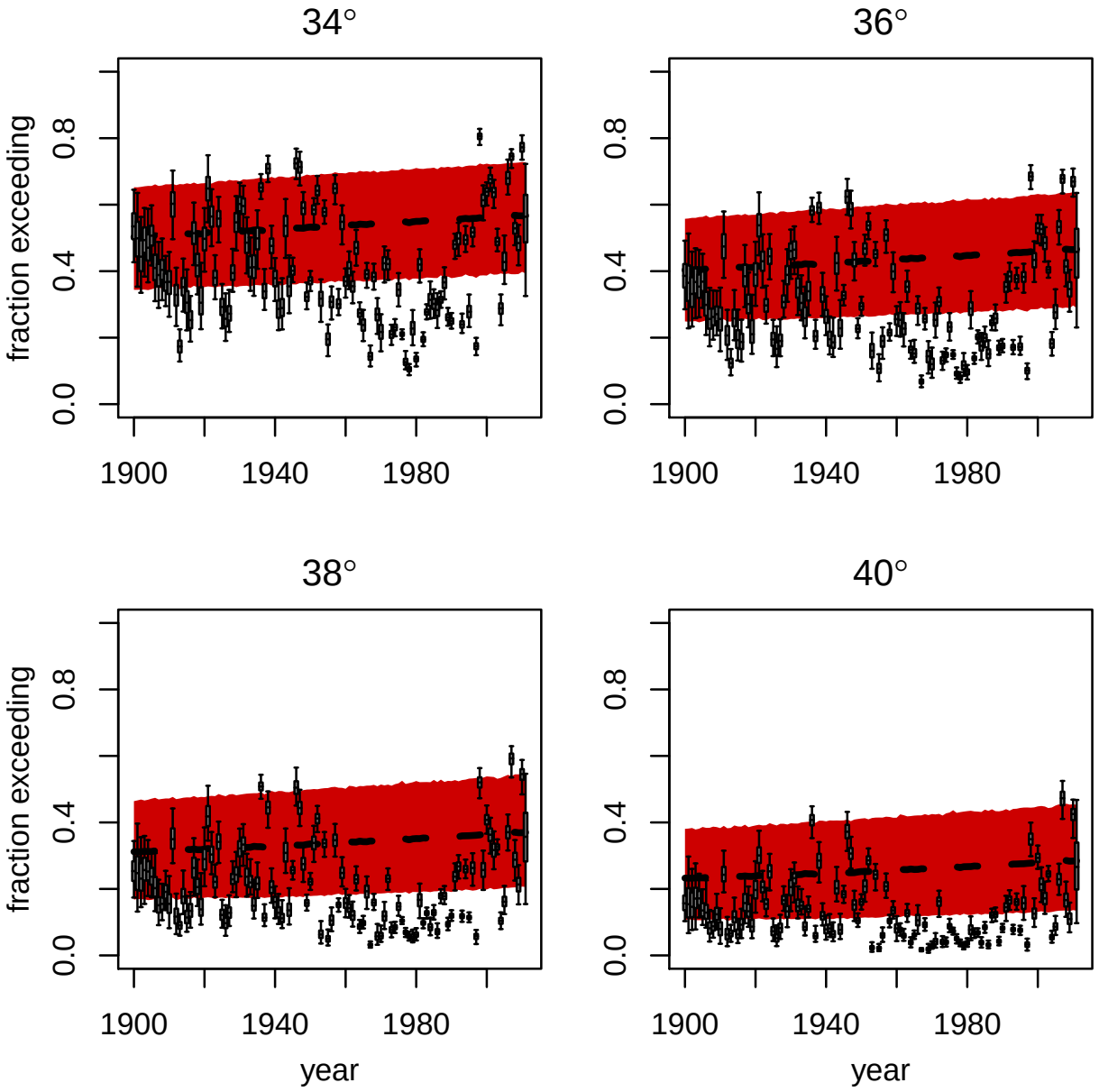


Prediction Locations

- Landcover data from MODIS MOD12 data product (`http://www-modis.bu.edu/landcover/`).
- Regular prediction grid on an equal area projection.



Fraction of Cropland Above Threshold



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Thoughts on the Future

1. Considering only annual maxes ignores valuable information.
   - Can use max-stable processes for threshold data (e.g. Jeon and Smith, 2013; Huser and Davison, 2013).
     - ? But how to use censored likelihoods?
   - Possible approach is Bortot and Gaetan (2013).
     - Won't work for temperatures.
     - Not as straightforward as I had hoped in the spatial context.
   - Generalized Pareto processes?

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2. Maybe asymptotic dependence is too strong.
   - Inverted max-stable processes (Wadsworth and Tawn, 2012).
     - Seems doable.

Summary

- The hierarchical framework produces spatial max-stable processes that have tractable likelihoods.

- By truncating and adding a PS(α) random effect, we managed to
  1. be Bayesian
  2. remain max-stable.

- This allows us
  1. modeling flexibility
  2. theoretical justification
  3. a platform for building more exotic max-stable models.

- We can do better!

The End

Thanks!

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