Penultimate Approximations: Past, Present ... and Future?*

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Abstract: The rate of convergence of the sequence of linearly normalized maxima or minima to the corresponding non-degenerate *extreme value* (EV) limiting distribution for maxima ($\mathrm{EVD_M}$) or for minima ($\mathrm{EVD_m}$) is a relevant problem in the field of *extreme value theory* (EVT). Fisher and Tippett (1928) observed that if we approximate the distribution of the suitably normalized sequence of maxima of normal samples not by the limiting distribution $G_0(x) = \exp(-\exp(-x)),\ x \in \Re$, but by a sequence of other EV distributions converging to G_0, the approximation is asymptotically improved. Such approximations are called *penultimate* or *pre-asymptotic approximations* and have been theoretically studied from different perspectives. The modern theory of rates of convergence in EVT began with Anderson(1971). Developments have followed several directions. For papers on the subject prior to 1992, see Gomes (1994). More recently, Gomes and de Haan (1999) derived, for all $\gamma \in \mathbb{R}$, exact penultimate approximation rates with respect to the variational distance, under von Mises-type conditions and some additional differentiability assumptions. For a recent short overview of the subject see Beirlant *et al.* (2012). Quite recently, this same topic has been revisited in the field of reliability, where any coherent system can be represented as either a series-parallel—a series structure with components connected in parallel— or a parallel-series system—a parallel structure with components connected in series (see Barlow and Proschan, 1975). Its lifetime can thus be written as the minimum of maxima or the maximum of minima. For large-scale coherent systems it can be sensible to assume that the number of system components goes to infinity. Then, the possible non-degenerate EV laws, $\mathrm{EVD_M}$ and $\mathrm{EVD_m}$, are eligible candidates for the finding of adequate lower and upper bounds for such a reliability. The identification of the possible limit laws for the system reliability of homogeneous series-parallel (or parallel-series) systems has already been done under different frameworks (see Reis and Canto e Castro, 2009, among others). However, as mentioned above, it is well-known that in most situations such non-degenerate limit laws are better approximated by an adequate penultimate distribution, being thus sensible to assess both theoretically and through Monte-Carlo simulations the gain in accuracy when a penultimate approximation is used instead of the ultimate one, as performed in Reis *et al.* (2012), for regular and homogeneous parallel-series systems, and in Gomes *et al.* (2013), for regular and homogeneous series-parallel systems. Moreover, researchers

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have essentially considered penultimate approximations in the class of $\mathrm{EVD_M}$ or $\mathrm{EVD_m}$, but we can easily consider a much broader scope for that type of approximations, and such a type of models surely deserves a deeper consideration under statistical backgrounds. *Penultimate models* seem to be possible and interesting alternatives to the classical models but have never been deeply used in the literature.

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