Extremes for deterministic and random dynamical systems

Jorge Milhazes Freitas
jmfreita@fc.up.pt
http://www.fc.up.pt/pessoas/jmfreita

Part of the work was developed with Ana Moreira Freitas and Mike Todd and another part with Hale Aytaç and Sandro Vaienti

CMUP
Centro de **Matemática**
Universidade do Porto

The setting

We will consider stochastic processes arising from dynamical systems (both deterministic and randomly perturbed systems). Namely, $X_0, X_1, X_2, \ldots$ will be such that:

$$X_n := \varphi \circ T^n = \varphi \circ \underbrace{T \circ \ldots \circ T}_{n \text{ times}}$$

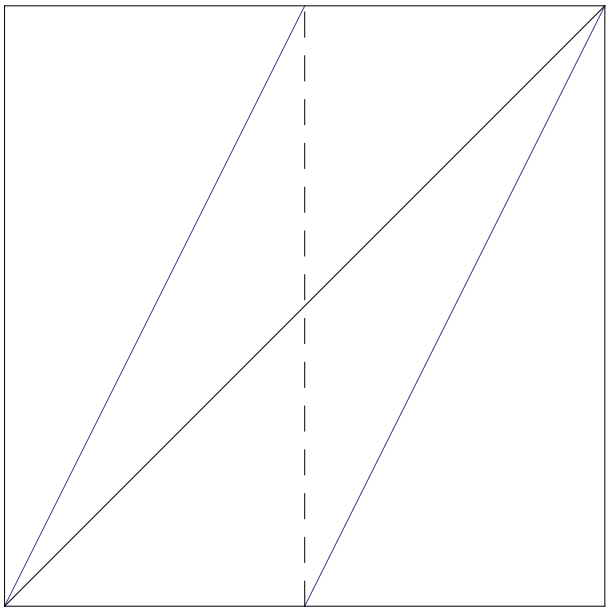
where the discrete time dynamical system $(\mathcal{X}, \mathcal{B}, \mathbb{P}, T)$ will denote two different but interrelated settings throughout the paper and $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, is just a measurable observable that achieves a global maximum at some $\zeta \in \mathcal{X}$.

$\mathcal{X}$ is a topological space, $\mathcal{B}$ is the Borel σ-algebra, $T : \mathcal{X} \to \mathcal{X}$ is a measurable map and $\mathbb{P}$ is a T-invariant probability measure, *i.e.*, $\mathbb{P}(T^{-1}(B)) = \mathbb{P}(B)$, for all $B \in \mathcal{B}$.

The T invariance of $\mathbb{P}$ implies that $X_0, X_1, \ldots$ is stationary.

Deterministic dynamics

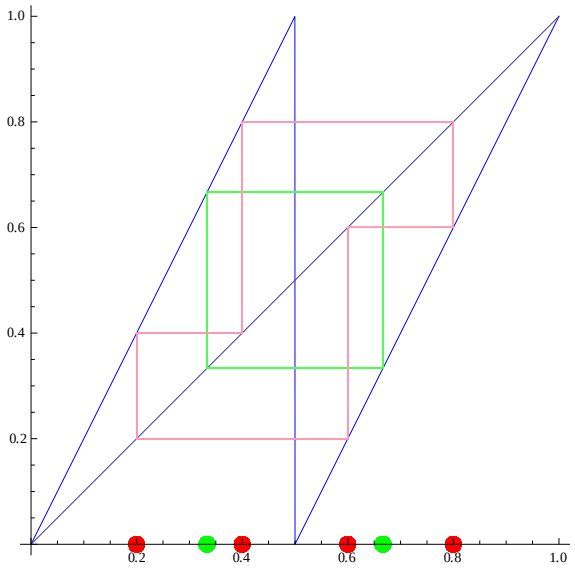
- $\mathcal{X} = \mathcal{M}$ is a compact Riemannian manifold
- $\mathcal{B}$ is the Borel σ-algebra
- $T = f : \mathcal{M} \to \mathcal{M}$ is a piecewise differentiable map
- $\mathbb{P} = \mu$ is an f-invariant probability
- Orbits: $x, f(x), f^2(x) := f(f(x)), \ldots$



$f : [0,1] \to [0,1]$
$f(x) = 2x \bmod 1$

Deterministic dynamics

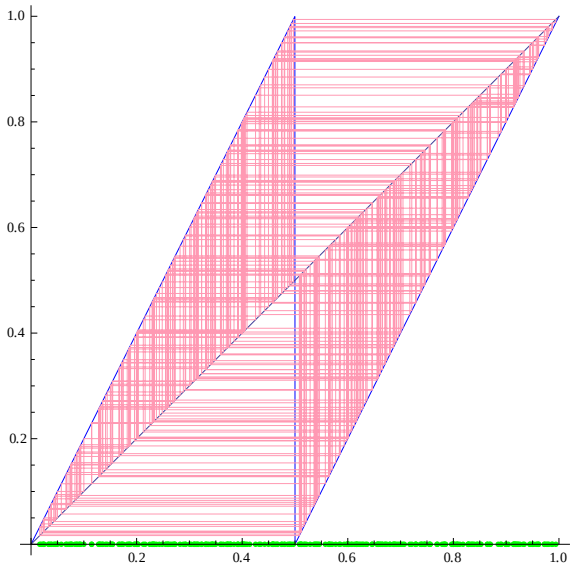
- $\mathcal{X} = \mathcal{M}$ is a compact Riemannian manifold
- $\mathcal{B}$ is the Borel σ-algebra
- $T = f : \mathcal{M} \to \mathcal{M}$ is a piecewise differentiable map
- $\mathbb{P} = \mu$ is an f-invariant probability
- Orbits: $x, f(x), f^2(x) := f(f(x)), \ldots$



$f : [0, 1] \to [0, 1]$
$f(x) = 2x \bmod 1$

Deterministic dynamics

- $\mathcal{X} = \mathcal{M}$ is a compact Riemannian manifold
- $\mathcal{B}$ is the Borel σ-algebra
- $T = f : \mathcal{M} \to \mathcal{M}$ is a piecewise differentiable map
- $\mathbb{P} = \mu$ is an f-invariant probability
- Orbits: $x, f(x), f^2(x) := f(f(x)), \ldots$



$f : [0, 1] \to [0, 1]$
$f(x) = 2x \bmod 1$

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \mathrm{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \mathrm{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let θ_ε be a probability supported on $B_\varepsilon(0) := \{y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon\}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb} \quad \text{and} \quad 0 < \underline{g_\varepsilon} \leq g_\varepsilon \leq \overline{g_\varepsilon} < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation f_ω by

$$f_\omega(x) = f(x) + \omega. \tag{1}$$

- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability θ_ε.
- Let $\Omega = B_\varepsilon(0)^{\mathbb{N}}$ and $\theta_\varepsilon^{\mathbb{N}}$ be the product measure defined on Ω
- Given a point $x \in \mathcal{M}$ and $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of x as $x, f_{\underline{\omega}}(x), f_{\underline{\omega}}^2(x), \ldots$ where:

$$f_{\underline{\omega}}^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),$$

with $f_{\underline{\omega}}^0$ being the identity map on $\mathcal{M}$.

Deterministic representation of random perturbations

Let

$$\begin{array}{rccc} S: & \mathcal{M} \times \Omega & \longrightarrow & \mathcal{M} \times \Omega \\ & (x, \underline{\omega}) & \longmapsto & (f_{\omega_1}(x), \sigma(\underline{\omega})), \end{array} \tag{2}$$

where $\sigma : \Omega \to \Omega$ is the shift $\sigma(\underline{\omega}) = \sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$.

Hence, the random evolution can fit the original model $(\mathcal{X}, \mathcal{B}, \mathbb{P}, T)$ by taking:

- $\mathcal{X} = \mathcal{M} \times \Omega$
- $\mathcal{B}$ is the respective product σ-algebra
- $\mathbb{P} = \mu_\varepsilon \times \theta_\epsilon^{\mathbb{N}}$
- the system is then given by the skew product map $T = S$.

Definition (Decay of correlations)

Let $\mathcal{C}_1, \mathcal{C}_2$ be two Banach spaces of real functions. We denote the *correlation* of non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ w.r.t. a measure $\mathbb{P}$ as

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1} \|\psi\|_{\mathcal{C}_2}} \left| \int \phi \, (\psi \circ T^n) \, \mathrm{d}\mathbb{P} - \int \phi \, \mathrm{d}\mathbb{P} \int \psi \, \mathrm{d}\mathbb{P} \right| .$$

We say that we have *decay of correlations*, w.r.t. $\mathbb{P}$, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if, for all $\phi \in \mathcal{C}_1$, $\psi \in \mathcal{C}_2$ we have

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) \to 0, \quad \text{as } n \to \infty.$$

We say that we have *decay of correlations against L^1 observables* whenever $\mathcal{C}_2 = L^1(\mathrm{Leb})$ and $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int |\psi| \, \mathrm{dLeb}$.

Stochastic processes arising from deterministic/random dynamical systems

Let $\varphi : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$.
In the deterministic setting $X_0, X_1, X_2, \ldots$ is given by

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

In the random dynamics case, the process will be

$$X_n = \varphi \circ f^n_{\underline{\omega}}, \quad \text{for each } n \in \mathbb{N}, \tag{4}$$

which can also be written as $X_n = \bar{\varphi} \circ S^n$, where $\bar{\varphi} : \mathcal{M} \times \Omega \to \mathbb{R} \cup \{+\infty\}$, is given by $\varphi(x, \underline{\omega}) = \varphi(x)$.
We assume that the $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ achieves a global maximum at $\zeta \in \mathcal{M}$ (we allow $\varphi(\zeta) = +\infty$).
We also assume that φ and $\mathbb{P}$ are sufficiently regular so that, for u sufficiently high, the event $U(u) = \{X_0 > u\}$ corresponds to a topological ball centred at ζ. Moreover, the quantity $\mathbb{P}(U(u))$, as a function of u, varies continuously on a neighbourhood of $u_F := \sup\{x : F(x) < 1\}$, where $F(x) = \mathbb{P}(X_0 \leq x)$.

Extreme Value Laws

We have an *exceedance* of the level $u \in \mathbb{R}$ at time $j \in \mathbb{N}$ if the event $\{X_j > u\}$ occurs.
Define a new sequence of random variables (r.v.) $M_1, M_2, \ldots$ given by

$$M_n = \max\{X_0, \ldots, X_{n-1}\}. \tag{5}$$

Definition

We say that we have an EVL for M_n if there is a d.f. $H : \mathbb{R} \to [0, 1]$, with $H(0) = 0$ and, for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$, s.t.

$$n\mathbb{P}(X_0 > u_n) \to \tau, \quad \text{as } n \to \infty, \tag{6}$$

and for which the following holds:

$$\mathbb{P}(M_n \leq u_n) \to \bar{H}(\tau), \quad \text{as } n \to \infty. \tag{7}$$

When $\bar{H}(\tau) = \mathrm{e}^{-\theta\tau}$, where $0 \leq \theta \leq 1$, we say we have an Extremal Index θ.

An extremal dichotomy

We begin with a dichotomy that was first realised to exist in [FFT12]. The actual statement we present here comes from the paper [AFV12] and uses the results of [FFT13] to cover the case of periodic points.

Theorem

Consider a continuous dynamical system $(\mathcal{M}, \mathcal{B}, \mu, f)$ *for which there exists a Banach space* $\mathcal{C}$ *such that for all* $\phi \in \mathcal{C}$ *and* $\psi \in L^1(\mu)$, $Cor_\mu(\phi, \psi, n) \leq Cn^{-2}$, *where* $C > 0$ *is a constant independent of both* ϕ, ψ. *Let* $X_0, X_1, \ldots$ *be given by* (3), *where* φ *achieves a global maximum at* ζ. *Let* u_n *be such that* (6) *holds. We assume that there exists* $C' > 0$ *such that for all* n *we have* $\mathbf{1}_{U(u_n)} \in \mathcal{C}$, $\|\mathbf{1}_{U(u_n)}\|_{\mathcal{C}} \leq C'$.

- *If* ζ *is a non-periodic point, then there exists an EVL for* M_n *and* $H(\tau) = 1 - \mathrm{e}^{-\tau}$.
- *If* ζ *is a repelling periodic point of prime period* p, *then there exists an EVL for* M_n *and* $H(\tau) = 1 - \mathrm{e}^{-\theta\tau}$, *where* $\theta = \lim_{n\to\infty} \mathbb{P}(X_0 > u_n, X_p \leq u_n | X_0 > u_n)$.

Smoothing effect of adding noise on the Extremal dichotomy

Theorem ([AFV12])

Consider a dynamical system $(\mathcal{M} \times \Omega, \mathcal{B}, \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}, S)$*, where* $\mathcal{M} = \mathbb{T}^d$*, for some* $d \in \mathbb{N}$*,* $f : \mathcal{M} \to \mathcal{M}$ *is a deterministic system which is randomly perturbed as in* (1) *and* S *is the skew product map defined in* (2)*. Assume that there exists* $\eta > 0$ *such that* $dist(f(x), f(y)) \leq \eta dist(x, y)$*, for all* $x, y \in \mathcal{M}$*. Assume also that the measure* μ_ε *is such that* $\mu_\varepsilon = h_\varepsilon \mathrm{Leb}$*, with* $0 < \underline{h}_\varepsilon \leq h_\varepsilon \leq \overline{h}_\varepsilon < \infty$*. Suppose that there exists a Banach space* $\mathcal{C}$ *of real valued functions defined on* $\mathcal{M}$ *such that for all* $\phi \in \mathcal{C}$ *and* $\psi \in L^1(\mu_\varepsilon)$*,* $Cor_{\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}}(\phi, \psi, n) \leq Cn^{-2}$*, where* $C > 0$ *is a constant independent of both* ϕ, ψ*.*
Let u_n *be such that* (6) *holds and assume that there exists* $C' > 0$ *such that for all* n *we have* $\mathbf{1}_{U(u_n)} \in \mathcal{C}$*,* $\|\mathbf{1}_{U(u_n)}\|_{\mathcal{C}} \leq C'$*. For any point* $\zeta \in \mathcal{M}$*, consider that* $X_0, X_1, \ldots$ *is defined as in* (4)*, then there exists an EVL for* M_n *such that* $H(\tau) = 1 - \mathrm{e}^{-\tau}$*.*

H. Aytaç, J. M. Freitas, and S. Vaienti, *Laws of rare events for deterministic and random dynamical systems*, Preprint arXiv:1207.5188 (2012).

A. C. M. Freitas, J. M. Freitas, and M. Todd, *The extremal index, hitting time statistics and periodicity*, Adv. Math. **231** (2012), no. 5, 2626 – 2665.

A. C. M. Freitas, J. M. Freitas, and M. Todd, *The compound poisson limit ruling periodic extreme behaviour of non-uniformly hyperbolic dynamics*, Comm. Math. Phys. **321** (2013), no. 2, 483–527.

Workshop on Risk Analysis and Extreme Values, Paris, 2005

Conference "Extreme Value Theory and Laws of Rare Events"

Centre International de Rencontres Mathématiques (CIRM)
Luminy, Marseille, France
From 14-07-2014 to 18-07-2014
http://www.cirm.univ-mrs.fr/index.html/

Organisers

- A.C.M. Freitas
- J.M. Freitas
- M. Todd
- S. Vaienti

Conference "Extreme Value Theory and Laws of Rare Events"

Centre International de Rencontres Mathématiques (CIRM)
Luminy, Marseille, France
From 14-07-2014 to 18-07-2014
http://www.cirm.univ-mrs.fr/index.html/

Scientific Committee

- P. Collet
- Michael Ghil
- Ivette Gomes
- Valerio Lucarini
- Matthew Nicol
- Holger Rootzén

Conference "Extreme Value Theory and Laws of Rare Events"

Centre International de Rencontres Mathématiques (CIRM)
Luminy, Marseille, France
From 14-07-2014 to 18-07-2014
http://www.cirm.univ-mrs.fr/index.html/

Scientific Committee

- P. Collet
- Michael Ghil
- Ivette Gomes
- Valerio Lucarini
- Matthew Nicol
- Holger Rootzén

Conference "Extreme Value Theory and Laws of Rare Events"

Centre International de Rencontres Mathématiques (CIRM)
Luminy, Marseille, France
From 14-07-2014 to 18-07-2014
http://www.cirm.univ-mrs.fr/index.html/

Scientific Committee

- P. Collet
- Michael Ghil
- Ivette Gomes
- Valerio Lucarini
- Matthew Nicol
- Holger Rootzén